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Time-asymmetric Fokker-type action and relativistic wave equations

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Abstract. A relativistic two-particle system with an arbitrary linear combination of scalar and vector time-asymmetric Fokker-type interactions in a two-dimensional spacetime is considered within the framework of the front form of dynamics. It is shown that the corresponding mass-shell equation takes the form of a linear relation between the generators of the Lie algebra $so(2,1)$. An algebraic quantization of the system is proposed and a closed form for the mass spectrum is obtained. The relativistic wave equation obtained by Barut and Rasmussen for the H atom is generalized to the case of an arbitrary linear combination of scalar and vector interactions. An extension of the results to the system in four-dimensional spacetime is suggested.

1. Introduction

More than 20 years ago, Barut and Rasmussen [1, 2] proposed a covariant wave equation describing the relativistic hydrogen atom—the system of two charged particles with arbitrary rest masses m_1 and m_2 under the influence of mutual electromagnetic interaction. The starting point of their treatment was the common group structure of both the non-relativistic two-body Coulomb problem and the Dirac spinor equation for the electron. In the two cases the same dynamical group $SO(2,4)$ arises. Using this analogy, the cited authors have postulated an infinite-component relativistic wave equation [1, 3–5] of the form

$$(\mathcal{J}_\mu P^\mu + B\Gamma_4 + D)|\tilde{\Psi}(P)\rangle = 0 \tag{1.1}$$

where $\mathcal{J}_\mu$ is the conserved matter current,

$$\mathcal{J}_\mu = A_1\Gamma_\mu + A_2 P_\mu + A_3\Gamma_4 P_\mu \tag{1.2}$$

$\mu = \overline{0,3}$, the operators Γ_μ and Γ_4 belong to the Lie algebra $so(2,4)$ of the conformal group $SO(2,4)$ and P_μ is the total momentum of the system. In the rest frame, where $P_\mu = (M,0,0,0)$, $P_\mu P^\mu = M^2 > 0$, equation (1.1) becomes

$$(A_1 M\Gamma_0 + A_2 M^2 + (A_3 M^2 + B)\Gamma_4 + D)|\Psi(M)\rangle = 0. \tag{1.3}$$

The numerical coefficients A_1, A_2, A_3, B and D entering equations (1.1) and (1.2) can be determined by comparison with corresponding non-relativistic and one-body Klein–Gordon calculations. For the case of electromagnetic interactions, which is the only case considered in the literature [1, 4], they turn out to be

$$A_1 = 1 \qquad A_3 = \frac{1}{2m_2} \qquad B = \frac{m_2^2 - m_1^2}{2m_2} \tag{1.4}$$

$$A_2 = \frac{\alpha}{2m_2} \qquad D = -\alpha\frac{m_1^2 + m_2^2}{2m_2} \tag{1.5}$$

where α denotes an interaction constant (a product of the charges).

Equation (1.3) can be solved in a purely algebraic manner that enables one to obtain the mass spectrum, bound and scattering states and even to describe the relativistic Coulomb scattering and relativistic photo-effect [1–5]. Many problems of non-relativistic and relativistic quantum mechanics can be transformed into the form (1.3) and treated similarly in an algebraic way [3, 4]. In all such treatments, the inner states of the system are determined by a three-dimensional subalgebra $so(2,1) = sl(2,\mathbb{R})$ of the Lie algebra $so(2,4)$. This subalgebra, which is the Lie algebra of the Lorentz group in 2 + 1 dimensions, contains the operators Γ_0 and Γ_4 entering equation (1.3).

Rather unexpectedly, the same algebraic structure has been discovered for the classical two-particle one-dimensional systems with Coulomb-like interactions within the framework of relativistic direct interaction theory [6]. Such a theory constitutes one more approach to the consistent investigation of relativistic composite systems without explicit introduction of field quantities with their own degrees of freedom [7–9]. Nevertheless, it is possible to consider relativistic direct interactions with certain field-theoretical counterparts. Such considerations are especially effective when the processes of emission or absorption of real (not virtual) quanta are absent or inessential. A well known example is provided by time-symmetric Fokker–Wheeler–Feynman electrodynamics [7]. In this theory the motion of a charged particle is determined by the (time-symmetric) half-sum of retarded and advanced fields that are produced by other particles of the system. This approach has been extended to the wide class of interactions within the terms of the Fokker-type action integrals [8–10]. The latter are also capable of describing time-asymmetric models. In such models (for example, for the two-particle system) the first particle moves in the advanced field of the second one, which in turn is subject to the retarded field of the first one. Such models were extensively investigated (mainly in the electromagnetic case) because they admitted an exact solution of the relativistic two-body problem in the two-dimensional spacetime [6, 11, 12] as well as in the four-dimensional one [13, 14]. The integrability of all such models has been demonstrated for the two cases in [15] and [16, 17], respectively, by using certain forms of relativistic dynamics. In recent years it has been demonstrated that Fokker-type action integrals can be applied for a semiphenomenological description of two-particle relativistic confinement models [18] and the two-body system with gravitational interaction [19]. The same integrals can be used for the covariant description of spinning particle systems with electromagnetic and scalar interactions [20].

In particular, it has been shown that the front form of dynamics in the two-dimensional spacetime is free from all the difficulties connected with the no-interaction theorem [15]. In this form of dynamics the Poincaré-invariance condition does not forbid the existence of usual interaction Lagrangians for the N-body problem, depending on the derivatives of the order not higher than the first in the terms of physical coordinates of the particles. The fulfilment of the mentioned conditions implies via Nöther's theorem the existence of corresponding integrals of motion which reduce the relativistic two-body problem to the quadratures. The same is true for the two-body problem in the four-dimensional spacetime, if the isotropic forms of dynamics are used [16, 17]. Such 'standard' Lagrangians contain expressions, which correspond to time-asymmetric interactions, mediated by linear massless relativistic fields of various tensor ranks.

Consequently, the Hamiltonian description can be obtained by means of the standard Legendre transformation with covariant (in a given form of dynamics) particle coordinates as canonical variables. This opens the way to quantization, but the complicated dependence of relativistic interaction Hamiltonians on the canonical coordinates and momenta faces us with a factor ordering difficulty as well as with certain subtle points of the relativistic coordinate representation. Fortunately, it can be observed that for a two-particle system in the two-dimensional spacetime the classical mass-shell equation takes the form of a linear relation between the generators of the algebra $so(2, 1)$. Demanding the preservation of that relation after quantization, we come to the wave equations of the form (1.3). In such a way, the relativistic quantum mechanical two-body problem is transcribed into algebraic language. Moreover, equation (1.3) contains all the coefficients, entering the full four-dimensional equation (1.1), and these coefficients are determined by the consideration in the two-dimensional spacetime. Therefore, we may establish the relativistic wave equation (1.1) for other interactions besides the electromagnetic one. The demonstration of this is the main purpose of this paper. It should be stressed that a desirable algebraic structure is not postulated on the outside. It arises in a natural way from the classical phase space consideration.

The paper is organized as follows. In the next section, we briefly review the Fokker-type action formalism and its relation with time-symmetric and time-asymmetric field interactions. We then introduce the concept of the front form of dynamics in the two-dimensional spacetime and outline the classical Hamiltonian description of the two-particle system on the line with linear superposition of the time-asymmetric scalar and vector interactions. The realization of phase space for such a system as orbits of the coadjoint representation of the Lie algebra $so(2, 1)$ is also pointed out. In section 3 we discuss the algebraic quantization of the system and represent some of its results. Section 4 is devoted to the four-dimensional generalization. Corresponding generators of the algebra $so(2, 1)$ are written out and mass spectra for an arbitrary linear superposition of scalar and vector interactions are obtained. A comparison with the results of quasirelativistic approximation is carried out. We also include our four-dimensional results in the framework of the Bakamjian–Thomas model [21], supplemented by a certain spacetime interpretation. Finally, we briefly discuss the results and make some comments about unsolved problems and perspectives.

2. Fokker-type action integrals

One of the most interesting and promising fields in the classical relativistic direct interaction theory is the formalism of Fokker-type action integrals [8–10]. The reason for its physical importance consists in the clear connection between this formalism and the classical field theory.

Let us consider an N-particle system described by the parametric equations of particle worldlines

$$\gamma_a : \mathbb{R} \to \mathbb{M}_4 \qquad \tau \mapsto x_a^\mu(\tau) \qquad a = \overline{1, N} \tag{2.1}$$

in the Minkowski spacetime $\mathbb{M}_4$ endowed with the metric $\|\eta_{\mu\nu}\| = \mathrm{diag}(1, -1, -1, -1)$. The most general Fokker-type action integral can be written in the form [10]

$$S = S_{\mathrm{f}} + S_{\mathrm{int}} = -\sum_{a=1}^{N} m_a \int \mathrm{d}\tau_a \sqrt{\dot{x}_a^2} - \sum\sum_{a<b} \int \mathrm{d}\tau_a \int \mathrm{d}\tau_b \sqrt{\dot{x}_a^2 \dot{x}_b^2}\,\Lambda_{ab}(\rho_{ab}, \omega_{ab}, \sigma_{ab}, \sigma_{ba}) \tag{2.2}$$

where $\dot{x}_a^\mu \equiv \mathrm{d}x_a^\mu(\tau_a)/\mathrm{d}\tau_a$ and Λ_{ab} are arbitrary Poincaré-invariant functions of the following scalar arguments:

$$\rho_{ab} = \eta_{\mu\nu}(x_a^\mu - x_b^\mu)(x_a^\nu - x_b^\nu) \qquad \omega_{ab} = \frac{\eta_{\mu\nu}\dot{x}_a^\mu \dot{x}_b^\mu}{\sqrt{\dot{x}_a^2 \dot{x}_b^2}} \qquad \sigma_{ab} = \frac{\eta_{\mu\nu}(x_a^\mu - x_b^\mu)\dot{x}_a^\nu}{\sqrt{\dot{x}_a^2}}. \tag{2.3}$$

The functions Λ_{ab} of interest in this paper are of the form

$$\Lambda_{ab} = g_a g_b F(\omega_{ab}) G(x_a - x_b) \tag{2.4}$$

where $g_a \in \mathbb{R}$ denote the coupling constants (particle 'charges'). If the function $G(x)$ is the Green function of the d'Alembert equation and $F(1) = 1$, then in the non-relativistic limit ($c \to \infty$) the action integral (2.2) gives rise to the Coulomb interaction potential. For instance, the functions

$$\Lambda_{ab} = e_a e_b \omega_{ab} \delta(\rho_{ab}) \tag{2.5}$$

lead to the Wheeler–Feynman electrodynamics [7]. In this case

$$G(x_a - x_b) = G(\rho_{ab}) = \delta(\rho_{ab}) \tag{2.6}$$

is the time-symmetric Green function of the d'Alembert equation.

The description based on the time-symmetric Green function (2.6) leads to a non-local (in time) Lagrangian function and, therefore, to the difference–differential equations of motion [10, 22]. The known Hamiltonization procedures for such equations (see [9, 22, 23]) use various approximations and are too complicated for immediate quantization. There exist simpler (but of course less realistic) models based on the time-asymmetric Green functions of the d'Alembert equation. For example, we can use in equation (2.4) the retarded Green function

$$G(x) = 2\Theta(x^0)\delta(x^2). \tag{2.7}$$

It gives the action (2.2) with

$$S_{\mathrm{int}} = -2\sum\sum_{a<b} g_a g_b \int \mathrm{d}\tau_a \int \mathrm{d}\tau_b \sqrt{\dot{x}_a^2 \dot{x}_b^2}\, F(\omega_{ab})\Theta(x_a^0 - x_b^0)\delta(\rho_{ab}) \tag{2.8}$$

which will be the main subject of our following consideration. If

$$F(\omega) = \omega^s \qquad s = 0, 1 \tag{2.9}$$

then the action (2.8) will correspond to the particle interaction through a local relativistic massless field of rank s in such a way that the ath particle responds only to the retarded field produced by particle b and the bth particle responds to the advanced field produced by particle a. Because of the antisymmetry in the relative time variable of the action integral (2.8) such theories are called *time asymmetric* [8, 10].

Reparametrization-invariance of Fokker's action (2.2) allows one to apply the useful notion of forms of relativistic dynamics [9], introduced by Dirac [24]. Using a time-asymmetric Green function one can obtain local single-time Lagrangians for the two-particle system in the four-dimensional Minkowski space by means of the lightcone form of dynamics [16, 17] as well as for an N-particle system in the two-dimensional spacetime by means of the front form of dynamics [15]. The latter case will be considered in the next section.

3. The front form of relativistic dynamics in $\mathbb{M}_2$ and scalar and vector time-asymmetric interactions

The front form of relativistic dynamics in the two-dimensional spacetime $\mathbb{M}_2$ with coordinates (x^0, x) corresponds to the foliation of $\mathbb{M}_2$ by isotropic hyperplanes [15]

$$x^0 + x = \tau. \tag{3.1}$$

This foliation defines the simultaneity relation between the events of the particle worldlines. The quantity τ plays the role of the evolution parameter of a system. The motion of particles is described by the functions $x_a(\tau)$ and the parametric equations (2.1) of the particle worldlines take the form

$$x = x_a(\tau) \qquad x^0 = \tau - x_a(\tau). \tag{3.2}$$

Within the framework of the Lagrangian formalism the functions $x_a(\tau)$ are determined by Hamilton's action principle $\delta S = 0$ with an action integral

$$S = \int \mathrm{d}\tau\, L. \tag{3.3}$$

The general structure of the Lagrangian function L follows from the Poincaré-invariance conditions. Fortunately, the family of simultaneity hyperplanes (3.1) is invariant with respect to the Poincaré group $\mathcal{P}(1, 1)$. This fact permits the existence of non-trivial interaction Lagrangians, which do not contain derivatives of orders higher than the first one. Such Lagrangians for an N-particle system can be written in the form [15]

$$L = -\sum_{a=1}^{N} m_a \gamma_a^{-1} + \sum\sum_{a<b} r_{ab} V_{ab}(r_{ab}\gamma_a, r_{ab}\gamma_b) \tag{3.4}$$

where $\gamma_a^{-1} \equiv \sqrt{1 - 2v_a}$, $v_a \equiv \mathrm{d}x_a/\mathrm{d}\tau$, $r_{ab} \equiv x_a - x_b$ and $V_{ab}(y_1, y_2)$ are arbitrary functions of the two indicated arguments.

As a consequence of the general properties of Lagrangian mechanics, the invariance under a three-parameter Poincaré group $\mathcal{P}(1, 1)$ leads to three conservation laws: of the energy E, of the momentum P and of the centre-of-inertia integral of motion K. They are given by [15]

$$E = \sum_{a=1}^{N} v_a \frac{\partial L}{\partial v_a} - L \qquad P = \sum_{a=1}^{N} \frac{\partial L}{\partial v_a} - E \qquad K = -\tau P + \sum_{a=1}^{N} x_a \frac{\partial L}{\partial v_a}. \tag{3.5}$$

It can be demonstrated that the Fokker-type action integral (2.8) in the front form of dynamics in $\mathbb{M}_2$ takes the form (3.3) with the Lagrangian [15]

$$L = -\sum_{a=1}^{N} m_a \gamma_a^{-1} - \sum\sum_{a<b} g_a g_b \frac{\gamma_a^{-1}\gamma_b^{-1}}{r_{ab}} F(\delta_{ab}) \qquad r_{ab} > 0 \tag{3.6}$$

where

$$\delta_{ab} \equiv \frac{1}{2}\left(\frac{\gamma_a}{\gamma_b} + \frac{\gamma_b}{\gamma_a}\right). \tag{3.7}$$

We note that the initial asymmetry with respect to the particle permutations in action (2.8) is reflected in the asymmetric conditions $r_{ab} > 0$, $a < b$. Expression (3.6) is obtained by means of a general scheme of transition from the Fokker-type action (2.2) to its single-time form described in [10]. Generally, this procedure leads to the Lagrangians with shifted time arguments of the particle coordinates. The crucial peculiarity of the front form in the two-dimensional spacetime consists in the fact that this shift can be got rid of (for details see [15]).

In the following, we shall consider only the two-particle system. The integrals of motion $P_\pm \equiv E \pm P$ corresponding to the Lagrangian (3.6) are given by

$$P_+ = m_1\gamma_1 + m_2\gamma_2 - \frac{\alpha}{|r|}B(\delta) \tag{3.8}$$

$$P_- = m_1\gamma_1^{-1} + m_2\gamma_2^{-1} \tag{3.9}$$

where

$$B(\delta) = 2\left(-\delta F(\delta) + (\delta^2 - 1)F'(\delta)\right) \tag{3.10}$$

$\alpha = g_1 g_2$, $r = r_{12}$, $\delta = \delta_{12}$ and F' denotes the first derivative with respect to the indicated argument. From equations (3.8) and (3.9) we obtain the total mass (inner energy) of the system:

$$M^2 = E^2 - P^2 = P_+P_- = (m_1^2 + m_2^2 + 2m_1m_2\delta)\left[1 + \frac{\alpha}{|r|P_+}B(\delta)\right]^{-1}. \tag{3.11}$$

The transition to the Hamiltonian description can be performed by means of the standard Legendre transformation. The canonical momenta are defined as

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial v_1} = m_1\gamma_1 - \frac{\alpha}{|r|}\left[-\frac{\gamma_1}{\gamma_2}F(\delta) + \frac{\gamma_1}{2\gamma_2}\left(\frac{\gamma_1}{\gamma_2} - \frac{\gamma_2}{\gamma_1}\right)F'(\delta)\right] \\ p_2 &= \frac{\partial L}{\partial v_2} = m_2\gamma_2 - \frac{\alpha}{|r|}\left[-\frac{\gamma_2}{\gamma_1}F(\delta) + \frac{\gamma_2}{2\gamma_1}\left(\frac{\gamma_2}{\gamma_1} - \frac{\gamma_1}{\gamma_2}\right)F'(\delta)\right]. \end{aligned} \tag{3.12}$$

Solving the system (3.12) with respect to velocities v_a and substituting the result into the expressions (3.5) for the conserved quantities, we obtain the canonical generators of the Poincaré group $\mathcal{P}(1,1)$. As can be easily demonstrated, after such substitutions the function (3.11) becomes $M^2 = M^2(rp_1, rp_2)$; it depends only on the Poincaré-invariant expressions rp_a and, therefore, has vanishing Poisson brackets with all the generators of the Poincaré group. These generators are then given by

$$\begin{aligned} &P_+ = p_1 + p_2 \qquad P_- = M^2/P_+ \\ &K = -\tau(P_+ + P_-)/2 + x_1p_1 + x_2p_2. \end{aligned} \tag{3.13}$$

They satisfy the following Poisson bracket relations:

$$\{P_+, P_-\} = 0 \qquad \{K, P_\pm\} = \pm P_\pm. \tag{3.14}$$

The system (3.12) was solved explicitly in the case (2.9) for the interactions mediated by the scalar ($s = 0$) or vector ($s = 1$) fields in the work [6]. Here we shall consider an arbitrary linear superposition of the scalar and vector interactions:

$$F(\delta) = \alpha_0 + \alpha_1\delta \tag{3.15}$$

where α_0 and α_1 are arbitrary dimensionless constants. Thus we obtain

$$M^2 = P_+\frac{m_1^2p_2 + m_2^2p_1 + \alpha(2m_1m_2\alpha_0 - (m_1^2 + m_2^2)\alpha_1)/|r|}{p_1p_2 + \alpha^2(\alpha_1^2 - \alpha_0^2)/|r|^2 - \alpha\alpha_1P_+/|r|}. \tag{3.16}$$

Although the existence of the proper non-relativistic limit demands $F(1) = 1$ and, therefore, $\alpha_0 + \alpha_1 = 1$, we shall not put this restriction over the main text of the paper. As was pointed out by Duviryak [25], the case $\alpha_0^2 = \alpha_1^2$ which includes $\alpha_0 + \alpha_1 = 0$, leads to a very interesting exactly solvable classical model of the relativistic two-particle interaction. The quantum counterpart of these models will be considered in the following sections.

The separation of external and internal motions is carried out by the choice P_+ and $R = K/P_+$ as new external canonical variables. As internal variables we choose (see [26])

$$\xi = \frac{m_2 p_1 - m_1 p_2}{P_+} \qquad q = r\frac{P_+}{m} \qquad \{q, \xi\} = 1 \tag{3.17}$$

where $m = m_1 + m_2$.

We note that even in the case of an arbitrary function $F(\delta)$ we can obtain the mass-shell equation

$$\Phi(q, \xi, M^2) = 0 \tag{3.18}$$

in the implicit (parametric) form. Indeed, inserting (3.12) into (3.17) and using (3.11), we find the relations

$$(\xi - \xi_0)^2 = \frac{4m_1^2 m_2^2 m^2(\delta^2 - 1)[F' - \mu(\delta F' - F)]^2}{M^4 B^2(\delta)} \tag{3.19}$$

$$q = \frac{\alpha m M^2 B(\delta)}{2m_1 m_2(\delta - \mu)} \tag{3.20}$$

where we have introduced the following constants of motion:

$$\xi_0 = \frac{(m_1 - m_2)(m^2 - M^2)}{2M^2} \qquad \mu = \frac{M^2 - m_1^2 - m_2^2}{2m_1 m_2}. \tag{3.21}$$

Equations (3.19) and (3.20) give the parametric representation of the mass shell (3.18) in terms of the parameter $\delta \in [1, \infty)$. Since sign(r) is an integral of motion [15], we shall consider only the case $r > 0$ ($q > 0$). The mass-shell equation (3.18) determines an inner motion of the system and allows one to integrate the classical two-body problem [27].

4. Algebraic realization of the mass-shell equation

We now return to the examination of the superposition of the scalar and vector interactions (3.15). Using in equation (3.16) the inner variables (3.17), we find the mass-shell equation in the form

$$Y(q, \xi)M^2 - X(q, \xi) = 0 \tag{4.1}$$

where

$$X = m\left(m m_1 m_2 q + m(m_2 - m_1)q\xi + \alpha(2m_1 m_2 \alpha_0 - (m_1^2 + m_2^2)\alpha_1)\right) \tag{4.2}$$

$$Y = m_1 m_2 q + (m_2 - m_1)q\xi - q\xi^2 + \alpha^2(\alpha_1^2 - \alpha_0^2)q^{-1} - \alpha\alpha_1 m. \tag{4.3}$$

Following Barut [4], we introduce three functions of canonical variables

$$J_0 = \frac{1}{2}\left(\beta q\xi^2 + \frac{q}{\beta} + \frac{\beta Q}{q}\right) \qquad J_1 = \frac{1}{2}\left(\beta q\xi^2 - \frac{q}{\beta} + \frac{\beta Q}{q}\right) \qquad J_2 = q\xi \tag{4.4}$$

where β and Q are arbitrary constants (the first of which is necessary only for dimensional reasons). The functions (4.4) span, under Poisson bracketing, the Lie algebra $so(2, 1)$:

$$\{J_0, J_1\} = J_2 \qquad \{J_1, J_2\} = -J_0 \qquad \{J_2, J_0\} = J_1 \tag{4.5}$$

and satisfy an identity

$$J_0^2 - J_1^2 - J_2^2 = Q. \tag{4.6}$$

The main point consists in the observation that putting

$$Q = \alpha^2(\alpha_0^2 - \alpha_1^2) \tag{4.7}$$

we can represent equation (4.1) into the form

$$J + C = 0 \tag{4.8}$$

where

$$J = aJ_0 + bJ_1 + dJ_2 \tag{4.9}$$

is an element of the Lie algebra $so(2,1)$ and the constants a, b, d and C are defined by

$$a = \frac{M^2}{\beta} + \beta m_1 m_2 (m^2 - M^2) \tag{4.10}$$

$$b = \frac{M^2}{\beta} - \beta m_1 m_2 (m^2 - M^2) \tag{4.11}$$

$$d = (m_2 - m_1)(m^2 - M^2) \tag{4.12}$$

$$C = 2\alpha m m_1 m_2 (\alpha_0 + \alpha_1 \mu). \tag{4.13}$$

In other words, the mass-shell equation (4.1) takes the form of a linear relation between the generators of the algebra $so(2,1)$. As we shall demonstrate in the next section, the existence of such an algebraic structure of the mass-shell equation plays a prominent role on the quantum level and enables one to obtain mass spectra without an explicit realization of the operators in a certain Hilbert space. Therefore, one can also expect that the algebra $so(2,1)$ could play an important role in the classical description.

Defining the Killing form of the element (4.9) as [28]

$$(J, J) = a^2 - b^2 - d^2 \tag{4.14}$$

we obtain

$$(J, J) = 4m^2 m_1^2 m_2^2 (1 - \mu^2). \tag{4.15}$$

A natural action of the Lorentz group $SO(2,1)$ on its Lie algebra by inner automorphisms preserves the quadratic form (4.14) and the sign of a. Following [28], the non-zero element $J \in so(2,1)$ will be said to be timelike, spacelike or lightlike, if $(J, J) > 0$, $(J, J) < 0$ or $(J, J) = 0$, respectively. It is clear that J_0 is timelike and J_1, J_2 are spacelike. The element (4.9) will be timelike, if $\mu^2 < 1$ ($|m_1 - m_2| < M < m$), spacelike, if $\mu^2 > 1$ ($M > m$ or $0 < M < |m_1 - m_2|$) and lightlike, if $\mu^2 = 1$ ($M = m$ or $M = |m_1 - m_2|$).

It is well known (the Kirillov–Konstant–Souriau theorem) [29] that each orbit of the coadjoint representation of the Lie algebra carries a natural symplectic structure. In our case, the orbits are submanifolds in $so^*(2,1) \sim so(2,1) \sim \mathbb{R}^3$ [30]. Depending on the value of Q, they are a one-sheeted hyperboloid

$$\mathcal{N}_- = \{(J_0, J_1, J_2) \in \mathbb{R}^3 \mid J_0^2 - J_1^2 - J_2^2 = Q\} \tag{4.16}$$

if $Q < 0$, two sheets of the two-sheeted hyperboloid

$$\mathcal{N}_+^\sigma = \{(J_0, J_1, J_2) \in \mathbb{R}^3 \mid J_0^2 - J_1^2 - J_2^2 = Q,\ \operatorname{sign} J_0 = \sigma\} \tag{4.17}$$

if $Q > 0$, two cones

$$\mathcal{N}_0^\sigma = \{(J_0, J_1, J_2) \in \mathbb{R}^3 \mid J_0^2 - J_1^2 - J_2^2 = 0,\ \operatorname{sign} J_0 = \sigma\} \tag{4.18}$$

and the origin of coordinates, if $Q = 0$. For the given value of Q, determined by (4.7), we can consider the corresponding manifolds (4.16)–(4.18) as a proper inner phase space $\mathcal{N}$ of the system.

Let us consider the closed 2-form [31]

$$\omega = \frac{J^0\,\mathrm{d}J^2 \wedge \mathrm{d}J^1 + J^1\,\mathrm{d}J^0 \wedge \mathrm{d}J^2 + J^2\,\mathrm{d}J^1 \wedge \mathrm{d}J^0}{J_0^2 - J_1^2 - J_2^2}. \tag{4.19}$$

The equations

$$J_0 = J_0 \qquad J_1 = \sqrt{J_0^2 - Q}\cos\varphi \qquad J_2 = \sqrt{J_0^2 - Q}\sin\varphi \tag{4.20}$$

where $0 \leqslant \varphi < 2\pi$, $J_0^2 > Q$, determine an immersion $\imath : \mathcal{N} \to so(2,1)$. The 2-form (4.19) is non-degenerate on the orbits and gives the symplectic form on $\mathcal{N}$:

$$\omega|_{\mathcal{N}} = \imath^*\omega = \mathrm{d}\varphi \wedge \mathrm{d}J_0. \tag{4.21}$$

Using $\mathcal{N}$ as the inner phase space for our system can be the key to the global classical description of the problem. This is especially important because the Legendre transformation (3.12) $\Lambda : \mathbb{R}^4 \to \mathbb{P}$, where $\mathbb{P}$ is a phase space of our problem, is not a global diffeomorphism. As one can see from (3.12), the momentum variables are not defined, if $r \to 0$ or $\gamma_a \to 0$ (the particle velocity reaches the speed of light). Thus, the particle momenta are only local canonical coordinates. In the same way, we can say that the inner canonical variables $q,\ \xi$ form a local chart in the inner phase space. As a result, the global classical evolution of the system cannot be described only in terms of particle canonical variables $p_1,\ p_2,\ x_1,\ x_2$ (or collective variables $R,\ P_+,\ q,\ \xi$) [27].

5. The quantization procedure

The standard approach to quantization of the considered classical problem consists in the establishing of the correspondence between the canonical generators (3.13) of the Poincaré group $\mathcal{P}(1,1)$ and some Hermitian operators, determining a unitary representation of the group in a certain Hilbert space. This determines the squared mass operator $\hat{M}^2$ of the system. The eigenvalue equation

$$\hat{M}^2\psi = M^2\psi \tag{5.1}$$

describes the stationary states of inner motion. This method has been used in the two-dimensional spacetime in the front form of dynamics for a number of simple systems [16, 32]. In these papers the Weyl quantization rule and the momentum representation in the Hilbert space $\mathcal{H}_N^F = \mathrm{L}^2(\mathbb{R}_+^N; \mathrm{d}\mu_N^F)$, $\mathrm{d}\mu_N^F = \prod_{a=1}^N \Theta(p_a)\,\mathrm{d}p_a/p_a$ have been used. However, many difficulties arise when one applies this quantization rule to particle systems with field-type interactions. The first of them is that the vector interaction violates the positivity condition for momentum variables $p_a \geqslant 0$, which is necessary for the quantum-mechanical front-form description in the momentum representation [26]. The second difficulty is the cumbersome form of the integral equations which follows from equations (3.13) and (3.16). Moreover, as has been shown in [33] for the example of relativistic oscillator-like interaction [32], such a quantization procedure is not unique. This is connected with the complicated dependence of the relativistic interaction potentials on the coordinates and momenta. To avoid these difficulties we require the quantization scheme to preserve the linear relation (4.8) between the generators of the group $SO(2, 1)$. Hence, we replace functions (4.4) with Hermitian operators obeying the commutation relations of the Lie algebra $so(2, 1)$,

$$[\hat{J}_0, \hat{J}_1] = \mathrm{i}\hat{J}_2 \qquad [\hat{J}_1, \hat{J}_2] = -\mathrm{i}\hat{J}_0 \qquad [\hat{J}_2, \hat{J}_0] = \mathrm{i}\hat{J}_1 \tag{5.2}$$

and obtain a quantum-mechanical equation

$$(\hat{J} + C)\psi = 0 \tag{5.3}$$

where

$$\hat{J} = a\hat{J}_0 + b\hat{J}_1 + d\hat{J}_2. \tag{5.4}$$

The constant (integral of motion) M^2, entering the coefficients a, b, d and C, is now considered as an eigenvalue of the operator $\hat{M}^2$.

The general structure of the mass spectrum can be found on the basis of the relations (5.2) and (5.3) without specifying the realization of the operators $\hat{J}_0$, $\hat{J}_1$ and $\hat{J}_2$. Operator $\hat{J}_0$ is a timelike element of the algebra $so(2, 1)$ and as a generator of compact subgroup $SO(2)$ has a discrete spectrum:

$$\hat{J}_0|n\rangle = n|n\rangle. \tag{5.5}$$

The operator $\hat{J}_1$ is spacelike and has a continuum spectrum:

$$\hat{J}_1|\lambda\rangle = \lambda|\lambda\rangle \qquad \lambda \in \mathbb{R}. \tag{5.6}$$

Moreover, every timelike element $so(2.1)$ of the form (5.4) has a discrete spectrum with eigenvalues $\mathrm{sign}(a)\sqrt{(J, J)}n$, where $n \in$ Spectrum (J_0). It follows from the general arguments of Bacry ([28], theorem 1, corollary) and will be demonstrated explicitly later. Similarly, the spacelike element (5.4) has a continuum spectrum $\sqrt{-(J, J)}\lambda$, $\lambda \in \mathbb{R}$. Therefore, the different elements of the algebra $so(2, 1)$ with the same Killing form (4.14) have the same spectrum.

Let us use the method developed by Barut and Rasmussen [1] (see also [3, 4]) to treat their equation for the relativistic H atom. First, we consider equation (5.3) with a timelike operator $\hat{J}$, $(J, J) > 0$ (i.e. if $|m_1 - m_2| < M < m$). If we perform the transformation

$$\psi = \mathrm{e}^{-\mathrm{i}\chi_1(\hat{J}_0-\hat{J}_1)}\mathrm{e}^{-\mathrm{i}\chi_2\hat{J}_2}\psi' \tag{5.7}$$

then a simple calculation shows that after the choice

$$\chi_1 = \frac{d}{a+b} \qquad \tanh\chi_2 = \frac{2b + d\chi_1}{2a - d\chi_1} \tag{5.8}$$

equation (5.3) takes the form

$$\left(\mathrm{sign}(a)\sqrt{(J, J)}\hat{J}_0 + C\right)\psi' = 0. \tag{5.9}$$

Using (4.15), (4.10) and (4.13), we obtain

$$\left(\sqrt{1-\mu^2}\hat{J}_0 + \alpha(\alpha_0 + \alpha_1\mu)\right)\psi' = 0. \tag{5.10}$$

Considering ψ' as an eigenstate $|n\rangle$ of the operator $\hat{J}_0$, we obtain the equation for the discrete eigenvalues M^2:

$$\sqrt{1-\mu^2}\,n + \alpha(\alpha_0 + \alpha_1\mu) = 0. \tag{5.11}$$

When $n > 0$, the solutions to this equation exist, if

$$\alpha(\alpha_0 + \alpha_1\mu) < 0. \tag{5.12}$$

These solutions have the form

$$(M_n^{\pm})^2 = m_1^2 + m_2^2 + 2m_1m_2 \frac{-\alpha^2\alpha_1\alpha_0 \pm n\sqrt{n^2 + \alpha^2(\alpha_1^2 - \alpha_0^2)}}{n^2 + \alpha^2\alpha_1^2}. \tag{5.13}$$

If $(J, J) < 0$ ($\mu^2 > 1$, i.e. $M > m$ or $M < |m_1 - m_2|$), we choose in equation (5.7)

$$\chi_1 = \frac{d}{a+b} \qquad \tanh\chi_2 = \frac{2a - d\chi_1}{2b + d\chi_1} \tag{5.14}$$

and obtain

$$(2mm_1m_2\sqrt{\mu^2 - 1}\hat{J}_1 + C)\psi' = 0. \tag{5.15}$$

Using (5.6), we find

$$(M_\lambda^{\pm})^2 = m_1^2 + m_2^2 + 2m_1m_2 \frac{-\alpha^2\alpha_1\alpha_0 \pm \lambda\sqrt{\lambda^2 - \alpha^2(\alpha_1^2 - \alpha_0^2)}}{\alpha^2\alpha_1^2 - \lambda^2}. \tag{5.16}$$

In the same way one can solve the non-relativistic three-dimensional quantum Coulomb problem (see [3, 4], where several other examples were considered). The recent paper [34] gives a similar algebraic treatment of the Coulomb problem in six dimensions.

The eigenvalue n of $\hat{J}_0$ is determined by the quantum Casimir invariant (4.6)

$$\hat{J}_0^2 - \hat{J}_1^2 - \hat{J}_2^2 = Q. \tag{5.17}$$

The classical Casimir invariant of our problem is determined by (4.7). Its value can be changed after quantization as a consequence of various possibilities of ordering non-commutative operators. The quantity Q is the only element of the theory which is undetermined within the framework of a purely algebraic description. As we shall demonstrate below the correspondence condition with the one-particle problem in an external field implies that Q has to preserve its classical value at the quantum level.

As we pointed out in the previous section, the classical value of Q (see (4.7)) can be positive, negative or zero. It depends on the type of interaction or, more precisely, on the values of the dimensionless constants α_0 and α_1. In the pure scalar case $\alpha_0 = 1$, $\alpha_1 = 0$ and $Q > 0$. In the pure vector case $\alpha_0 = 0$, $\alpha_1 = 1$ and $Q < 0$. Therefore, the interaction with $Q > 0$ ($\alpha_0^2 > \alpha_1^2$) we shall call the scalar-type interaction. Interaction with $Q < 0$ ($\alpha_0^2 < \alpha_1^2$) we shall call the vector-type interaction. Let us put $\varphi = (-1 + \sqrt{1 + 4Q})/2$ so that $Q = \varphi(\varphi + 1)$. As a result, the scalar-type interaction leads to the discrete class $D_+(\varphi)$ of unitary irreducible representations of the group $SO(2, 1)$ [3, 4, 35]. In the case of the vector-type interaction we come to the supplementary class $D_+(\varphi, \varphi)$. Thus, if $Q > -\frac{1}{4}$, then

$$n = \varphi + k = (-1 + \sqrt{1 + 4Q})/2 + k \qquad k = 1, 2, \ldots. \tag{5.18}$$

In the case of a special linear superposition of the scalar and vector interactions, when $\alpha_0^2 = \alpha_1^2$ ($Q = 0$), we come to the discrete class $D_+(0)$ and eigenvalues of $\hat{J}_0$ are integer: $n = k$.

Let us consider a few particular cases in more detail. In the case of pure scalar or vector interactions (2.9) the mass spectrum takes the form

$$(M_n^{\pm})_s^2 = m_1^2 + m_2^2 \pm 2m_1m_2\big(1 - (-1)^s\alpha^2/n^2\big)^{(-1)^s/2}. \tag{5.19}$$

For the vector interaction the branch $(M_n^+)_1^2$ corresponds to attraction ($\alpha < 0$) and the branch $(M_n^-)_1^2$ corresponds to repulsion ($\alpha > 0$). For the scalar interaction both branches $(M_n^\pm)_0^2$ correspond to the attraction. Only the branch $(M_{n,\lambda}^+)_s^2$ has the correct non-relativistic limit.

In the one-particle limit ($m_1/m_2 \to 0$) we obtain from (5.19)

$$E = m_1\big(1 - (-1)^s\alpha^2/n^2\big)^{(-1)^s/2} - m_1 \tag{5.20}$$

that agrees with the one-particle spectrum for the Klein–Gordon problem with the scalar or vector Coulomb potential for the states with a zero value of the orbital quantum number.

In the case of the vector-type interaction, the expression (5.18) for the quantum number n has a sense only if the inequality $-\frac{1}{4} < Q < 0$ is satisfied, and we can also consider the state with $k = 0$. For the pure vector interaction the inner energy of this ground state is

$$(M_\varphi^\pm)_1^2 = m_1^2 + m_2^2 \pm 2m_1m_2\sqrt{-\varphi}\,. \tag{5.21}$$

The quantity $(M_\varphi^+)_1$ tends to infinity in the non-relativistic limit [36] and for the one-body problem gives the expression

$$m_1 + E = m_1\left[\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1-4\alpha^2}\right]^{1/2} \simeq m_1|\alpha|\left(1 + \tfrac{1}{2}\alpha^2 + \cdots\right) \tag{5.22}$$

which agrees with [37, 38]. The existence of such a strongly bounded ground state is typical for a one-dimensional electromagnetic interaction.

Equation (5.19) describes the mass spectra of two-particle systems with time-asymmetric scalar and vector interactions. The spectra of vector type ($s = 1$) have been obtained by Staruszkiewicz in the two-dimensional spacetime [13] and by Barut and Rasmussen in four-dimensional Minkowski space on the basis of an infinite-component wave equation (1.1) [1, 4, 5]. The two-particle bound state mass spectrum $(M_n^+)_0$ for scalar interaction in the case of equal particle masses $m_1 = m_2 = m/2$ turns out to be

$$(M_n^+)_0 = m\sqrt{\frac{1}{2}\left(1 + \sqrt{1 - \left(\frac{\alpha}{n}\right)^2}\right)}. \tag{5.23}$$

It has the same form as the mass spectrum obtained by Darewych within the framework of the reduced scalar Yukawa theory in [39]. The only difference lies in the definition of the quantum number n. In contrast to (5.18), in the mentioned work the quantum number n takes integer values: $n = k$.

In the case of a special linear superposition of the vector and scalar interactions $\alpha_0 = \frac{1}{2}\varepsilon$, $\alpha_1 = \frac{1}{2}$; $\varepsilon^2 = 1$ ($Q = 0$) the mass spectrum (5.13) becomes

$$(M_n^\varepsilon)^2 = m_1^2 + m_2^2 + 2m_1m_2\varepsilon\frac{4n^2 - \alpha^2}{4n^2 + \alpha^2} \qquad \alpha\varepsilon < 0. \tag{5.24}$$

In the case of half-difference ($\varepsilon = -1$) the discrete spectrum exists only for particles with the same signs of 'vector charges' and different signs of 'scalar charges'. In the non-relativistic limit the difference between vector and scalar interactions tends to zero and the system becomes free. The mass spectrum (5.24) for $\varepsilon = -1$ does not have any non-relativistic counterpart. The possibility of the existence of such essentially relativistic bound states for repulsion is typical not only for field-type interactions (see, for example, [32]). In the case of a half-sum ($\varepsilon = 1$) the mass spectrum (5.24) corresponds only to attraction and has a good non-relativistic limit—the Coulomb energy spectrum. In the one-particle limit it gives

$$E = m_1\frac{4n^2 - \alpha^2}{4n^2 + \alpha^2} \tag{5.25}$$

that agrees with the one-particle Klein–Gordon problem with the half-sum of the scalar and vector Coulomb potential for the states with zero value of the orbital quantum number [3].

6. Four-dimensional generalizations

The different elements of the algebra $so(2, 1)$ with the same Killing form (4.14) have the same spectrum. And vice versa, if different equations of the form (5.3) lead to the same mass spectrum then they contain elements of $so(2, 1)$ with the same Killing form which are related via the Lorentz transformation. Let us put

$$\psi = \mathrm{e}^{-\mathrm{i}\chi_1(\hat{J}_0-\hat{J}_1)}\mathrm{e}^{-\mathrm{i}(\chi_2-\Theta_B)\hat{J}_2}|\Psi(M)\rangle \tag{6.1}$$

in equation (5.3). If we choose

$$\tanh\Theta_B = \frac{M^2+m_2^2-m_1^2}{2Mm_2}. \tag{6.2}$$

and parameters χ_1, χ_2 defined by equation (5.8), then we arrive at wave equation (1.3),

$$\big(A_1MJ_0 + (A_3M^2+B)J_1 + A_2M^2 + D\big)|\Psi(M)\rangle = 0 \tag{6.3}$$

postulated by Barut and Rasmussen [1, 2] with the coefficients A_1, A_3, B given by (1.4) and

$$A_2 = \frac{\alpha\alpha_1}{2m_2} \qquad D = \alpha\left(\alpha_0 m_1 - \alpha_1\frac{m_1^2+m_2^2}{2m_2}\right). \tag{6.4}$$

The only difference between equations (6.3) and (1.3) consists in the different realizations of the Lie algebra $so(2, 1)$ in both equations. We denote that using a different notation for the elements of the basis of the Lie algebra $so(2, 1)$, i.e. Γ_0, Γ_4 in equation (1.3) and J_0, J_1 in our case, respectively.

Therefore, we can postulate the relativistic four-dimensional wave equation of the form (1.1) for the considered case of an arbitrary linear superposition of scalar and vector interactions. It is easy to see that the coefficients in equation (6.4) for the pure vector (electromagnetic) interaction ($\alpha_0 = 0$, $\alpha_1 = 1$) take the values given by (1.4) and (1.5).

It is necessary to point out that we do not postulate the coefficients (4.10)–(4.13). We obtain the classical mass-shell equation (4.8) and therefore its quantum counterpart (5.3) as a consequence of the Legendre transformation for the Lagrangian (3.6) which is connected with the Fokker-type action integral. Thus, spectrum (5.19) of the vector type can be obtained in our approach by an immediate quantization of the time-asymmetric electromagnetic interaction in the framework of the Hamiltonian description of a directly interacting particle system in the front form of dynamics.

One of the possible canonical realizations of the Lie algebra $so(2, 1)$ in some six-dimensional phase space was indicated by Barut [4]:

$$J_0 = \frac{1}{2}\left[\beta q\xi^2 + \frac{1}{\beta}q + \frac{\beta Q}{q}\right] \qquad J_1 = \frac{1}{2}\left[\beta q\xi^2 - \frac{1}{\beta}q + \frac{\beta Q}{q}\right] \qquad J_2 = (\boldsymbol{q}\cdot\boldsymbol{\xi}). \tag{6.5}$$

Here $q = \sqrt{q_1^2+q_2^2+q_3^2}$, $\boldsymbol{\xi}^2 = \xi_1^2+\xi_2^2+\xi_3^2$, $\{q_k,\ \xi_j\} = \delta_{kj}$ and $(\boldsymbol{q}\cdot\boldsymbol{\xi}) = q_1\xi_1+q_2\xi_2+q_3\xi_3$ is the Euclidean scalar product. The classical Casimir invariant has the form

$$J_0^2 - J_1^2 - J_2^2 = Q + \boldsymbol{L}^2 \tag{6.6}$$

where $\boldsymbol{L} = \boldsymbol{q}\times\boldsymbol{\xi}$. We assume that this relation is preserved after quantization and postulate the quantum wave equation in the form (1.3) with the coefficients (1.4) and (6.4). Then, considering the common eigenstates $|n, \ell\rangle$ of the operators J_0 and $\hat{\boldsymbol{L}}^2$ with $\hat{\boldsymbol{L}}^2|n, \ell\rangle = \ell(\ell+1)|n, \ell\rangle$, we obtain in full accordance with the previous case, the mass spectra in the form (5.13), where

$$\begin{aligned} n &= -\tfrac{1}{2} + k + \sqrt{\left(\ell+\tfrac{1}{2}\right)^2 + \alpha^2(\alpha_0^2-\alpha_1^2)} \\ \ell &= 0, 1, \ldots, k-1 \qquad k = 1, 2, \ldots. \end{aligned} \tag{6.7}$$

The expansion up to order c^{-2} gives the following correction to the energy spectrum of linear superposition of scalar and vector interactions for the branch $(M_n^+)_k^2$:

$$E \approx -\frac{m_1 m_2 \alpha^2}{m n_0^2 \hbar^2} \frac{(\alpha_0 + \alpha_1)^2}{2} - \frac{\alpha^4 (\alpha_0 + \alpha_1)^3 m_1 m_2}{4 m \hbar^4 n_0^4 c^2} \times \left[\left(1 + \frac{m_1 m_2}{m^2}\right) \frac{\alpha_0 + \alpha_1}{2} - 2\alpha_1 - \frac{4 n_0 (\alpha_0 - \alpha_1)}{2\ell + 1} \right] \tag{6.8}$$

where $n_0 = k + \ell$. When $\alpha_0 + \alpha_1 = 1$, the last formula corresponds to the energy spectra obtained in [40] for field-type interactions by means of the approximate relativistic Lagrangians of order c^{-2}. In the one-particle limit $(m_1/m_2 \to 0)$ we obtain expression (5.20) with the quantum number n given by (6.7), which agrees with the spectrum of the spinless particle in the external scalar or vector field.

Thus, equations (5.13) and (6.7) describe a quite profound mass spectrum. It is worthwhile constructing a relativistic two-particle Hamiltonian description which can lead to the mass spectrum (5.13) and (6.7). Of course such an 'inverse problem' is very ambiguous. However, our previous consideration (sections 3 and 4) suggests a simple solution as follows. Let us consider equation (4.8) with the generators J_0, J_1, J_2 of the form (6.5) as a classical mass-shell equation of the corresponding two-particle problem in $\mathbb{M}_4$. Then on the quantum level we obtain equations (5.3), (5.5), (6.7) and consequently we obtain the mass spectrum (5.13). However, these quantum equations do not provide the complete quantum relativistic description. They correspond only to an inner quantum dynamics and do not describe time evolution of external degrees of freedom. The inner quantum dynamics must be supplemented by a quantum relativistic description of the system as a whole and a unitary representation of the Poincaré group $\mathcal{P}(3, 1)$. The first and necessary step to include our results into a complete quantum relativistic Hamiltonian picture in the four-dimensional Minkowski space $\mathbb{M}_4$ is the solution to the similar problem on the classical level.

Our construction is based on a canonical realization (6.5) and the mass-shell equation in the form (4.8). On the other hand, we can express the squared total mass function in terms of the generators

$$M^2 = \frac{X(J_0, J_1, J_2)}{Y(J_0, J_1, J_2)} \tag{6.9}$$

where

$$\begin{aligned} X &= m\big(m m_1 m_2 \beta (J_0 - J_1) + m(m_2 - m_1) J_2 + \alpha(2 m_1 m_2 \alpha_0 - (m_1^2 + m_2^2)\alpha_1)\big) \\ Y &= (m_1 m_2 \beta - 1/\beta) J_0 - (m_1 m_2 \beta + 1/\beta) J_1 + (m_2 - m_1) J_2 - \alpha \alpha_1 m. \end{aligned} \tag{6.10}$$

Equation (6.9) can describe the inner motion of relativistic two-particle system in six-dimensional phase space in terms of canonical variables $(\boldsymbol{q}, \boldsymbol{\xi})$. To obtain the complete classical relativistic description of the two-body system it is necessary to construct ten canonical generators of the Poincaré group $\mathcal{P}(3, 1)$, which commute with the total mass of the system (6.9), and find relations between two sets of variables—canonical variables entering the canonical realization of $\mathcal{P}(3, 1)$ and covariant particle coordinates.

It is well known that there are a lot of possibilities to introduce an interaction into canonical generators of $\mathcal{P}(3, 1)$, which specifies stability group G_Σ—the subgroup of $\mathcal{P}(3, 1)$ containing generators independent of the interaction. The group G_Σ determines the Hamiltonian form of the dynamics. For a given total mass function all the Hamiltonian forms of dynamics are canonically equivalent [41]. Therefore, taking into account this fact we can easily carry out the first part of the programme. Indeed, let us consider well known canonical generators of the Poincaré group of the Bakamjian–Thomas model [21]:

$$\begin{aligned} &H = P_0 = \sqrt{M^2 + \boldsymbol{P}^2} \qquad \boldsymbol{P} = \boldsymbol{P} \\ &\tilde{\boldsymbol{J}} = \boldsymbol{R} \times \boldsymbol{P} + \boldsymbol{S} \\ &\boldsymbol{K} = -t\boldsymbol{P} + \boldsymbol{R}H + \frac{\boldsymbol{P} \times \boldsymbol{S}}{H + M}. \end{aligned} \tag{6.11}$$

Here $\boldsymbol{S}$ and M depend only on inner canonical variables and correspond, respectively, to the total spin and total mass of the system. The choice of the total mass function M (which can be an arbitrary function of inner variables) completely determines the Hamiltonian description.

Let variables $\boldsymbol{P}$, $\boldsymbol{R}$ entering canonical generators (6.11) of the Poincaré group, be external canonical variables of our problem, such that

$$\{R_j, q_i\} = \{R_j, \xi_i\} = \{P_j, q_i\} = \{P_j, \xi_i\} = 0 \tag{6.12}$$

and $\boldsymbol{S} = \boldsymbol{L}$. Then, choosing the total mass function in the form (6.9), ten functions (6.11) will satisfy commutation relations of the Poincaré algebra $\mathfrak{p}(3, 1)$ i.e. determine the canonical realization of the Poincaré group for our system. Consequently, of (6.12), generators (6.5) commute with external canonical variables $\boldsymbol{P}$, $\boldsymbol{R}$. Furthermore, $\{J_i, \tilde{J}_k\} = 0$. However, J_0, J_1, J_2 do not form any closed algebraic relations with other generators (6.11) of the Poincaré algebra. In such a manner the Lie algebra generated by (6.5) appears only as the algebra of the inner dynamical group $SO(2, 1)$. It is worth noting that the mass-shell equation (4.8) with the generators of $so(2, 1)$ in the form (6.5) fits into the general structure of the mass-shell constraint provided that a certain gauge fixing is used in the manifestly covariant Hamiltonian mechanics on the lightcone [17].

To solve the second task we have to supplement the Bakamjian–Thomas model description by a spacetime interpretation. The most general solution to the problem of construction of the covariant particle coordinates in terms of canonical variables was found by Duviryak and Kluchkovsky in [42]. These authors have proved that for a given total mass function (which completely determines the Hamiltonian dynamics) one can construct a great deal of worldlines in $\mathbb{M}_4$. This is due to the dependence of the solution to the problem (in the two-particle case) on six scalar functions of inner canonical variables. Therefore, starting from the canonical realization (6.11) of the Poincaré group with total mass function (6.9) we cannot reconstruct the total picture in $\mathbb{M}_4$ without ambiguities. However, we can restrict the ambiguity as described below. In the two-dimensional spacetime we have immediately obtained the classical Hamiltonian description of the two-particle system with the superposition of scalar and vector interactions from the time-asymmetric Fokker-type action integral. We cannot be sure that this is true in $\mathbb{M}_4$ for the total mass function (6.9) with arbitrary values of the coupling constants α_0, α_1. However, one can show that in $\mathbb{M}_4$ there exists such a relation between the Hamiltonian description with the total mass (6.9) and the time-asymmetric Fokker action at least for the scalar–vector interaction ($\alpha_0^2 = \alpha_1^2$, $Q = 0$). Indeed, in this case the classical mass-shell equation obtained in [25] within the framework of manifestly covariant Hamiltonian description in $T^*\mathbb{M}_4^2$ after a certain gauge fixing [17], can be put in the form

$$J + C = \tilde{a}J_0 + \tilde{b}J_1 + \tilde{d}J_2 + C = 0 \tag{6.13}$$

where

$$\tilde{a} = \left(\frac{1}{\beta} - \beta\frac{m_1^2 m_2^2(\mu^2 - 1)}{M^2}\right) mM \qquad \tilde{d} = 0$$
$$\tilde{b} = -\left(\frac{1}{\beta} + \beta\frac{m_1^2 m_2^2(\mu^2 - 1)}{M^2}\right) mM. \tag{6.14}$$

The generators J_0, J_1 and J_2 of the Lie algebra $so(2,1)$ are scalar functions of some inner canonical variables $\boldsymbol{k}$, $\boldsymbol{z}$. In terms of variables $\boldsymbol{\xi}$, $\boldsymbol{q}$ they have the same form as generators (6.5) with $Q = 0$.

The Killing form of the element $\tilde{J}$ of the Lie algebra $so(2,1)$ in equation (6.13) is equal to (4.15). Thus, using a quantization method such as that above, we obtain a quantum equation in the form (5.3) and as a result we obtain the mass spectrum (5.24) with the quantum number $n = k + l$. The problem of the spacetime interpretation of the Hamiltonian description in this case is solved due to its immediate relationship with the Fokker-type action integral [17, 25].

One can show that there exists a transformation $(\boldsymbol{k}, \boldsymbol{z}) \mapsto (\boldsymbol{\xi}, \boldsymbol{q})$ which transforms equation (6.13) into equation (4.8). We now assume that the relations between the covariant and the canonical variables are of the same form for arbitrary α_0, α_1. Then, using the results of [17], we obtain

$$x_a^\mu = X^\mu + \left[\Lambda^{\mathrm{T}}(P/M)\right]^\mu{}_\nu e_a^\nu(\boldsymbol{\xi}, \boldsymbol{q}) \tag{6.15}$$

where

$$e_a^0 = \frac{1}{M}\left((-)^{\bar{a}}\frac{m}{2}q - \frac{(m_2 - m_1)(M^2 + 2m^2)}{2M^2}q + (\boldsymbol{\xi}\cdot\boldsymbol{q})\right) \qquad a = 1, 2$$
$$e_a^i = \frac{1}{M}\left((-)^{\bar{a}}\frac{m}{2}q^i - \frac{(m_2 - m_1)(M^2 - 2m^2)}{2M^2}q^i + q\xi^i\right) \qquad \bar{a} = 3 - a \tag{6.16}$$

and

$$X^0 = t \tag{6.17}$$

$$X^i = R^i - \frac{(\boldsymbol{P}\times\boldsymbol{S})^i}{M(M + H)} \tag{6.18}$$

are the well known Pryce centre-of-inertia variables. The matrix

$$\|\Lambda^\mu{}_\nu\| = \left\|\begin{matrix} \dfrac{P_0}{|P|} & \dfrac{P_j}{|P|} \\ \dfrac{P_i}{|P|} & \delta_{ij} + \dfrac{P_i P_j}{|P|(|P| + P_0)} \end{matrix}\right\| \tag{6.19}$$

describes a pure boost transformation into the centre-of-inertia reference frame [17]. Equations (6.15)–(6.19) correspond to the special choice of general expressions for the covariant coordinates proposed in [42]. Using expressions (6.11) for the generators of the Poincaré algebra it is easy to verify that quantities (6.15) have proper transformation properties, i.e. they really correspond to covariant particle coordinates.

7. Conclusions

The traditional point of view on the relativistic few-body problem is based on quantum field theory. Mostly, wave equations describing few-body states are derived from Bethe–Salpeter-type equations via various reduction schemes. Unfortunately, this method is not free from difficulties and ambiguities. Indeed, as is well known [43–45]: (a) Bethe–Salpeter equations have unphysical solutions; (b) the relative time variable plays a dynamical role; (c) perturbation theory must be used to obtain the corresponding kernels; (d) the approach is hard to apply for more than two particles. Even the recent attempts to put the Bethe–Salpeter equation on rigorous mathematical grounds [46] crucially depends on the rigorous construction of the quantum field theory which is absent in $\mathbb{M}_4$ (see, e.g., [47]). Moreover, the relativistic quantum mechanical description obtained from the Bethe–Salpeter-type equation does not have, in general, a direct spacetime interpretation [48].

Therefore, it could be suitable to accommodate the idea that starting from classical field theory we may avoid some of the difficulties in the description of the relativistic few-body problem. The natural link between classical field theory and relativistic particle mechanics is given by the formalism of Fokker-type action integrals. One of the most important features of the Fokker-type action integrals is an obvious spacetime interpretation of the corresponding relativistic models, which are related to a field-theoretical description of particle interactions. It is preserved for two-particle systems with time-asymmetric interactions after transition to the Hamiltonian description. The preservation of such a feature after further quantization could be the key to the solution of many problems.

In this paper we have considered a two-particle system with Coulomb-like interactions which are mediated in the two-dimensional Minkowski space by massless relativistic fields in a time-asymmetric manner. This corresponds to the choice of the time-asymmetric Green function of the d'Alembert equation and means that the first particle responds only to a retarded field and the second particle responds to an advanced field.

Using such a Poincaré-invariant model, we avoid all difficulties related to non-locality of the description based on the Fokker-type action integrals. Our system is described by the Lagrangian function depending only on covariant particle coordinates and velocities. This Lagrangian is the starting point for our studies in the present paper.

The two-particle Lagrangian (3.6) with the interaction function $F(\delta) = \alpha_0 + \alpha_1\delta$ describing the superposition of the scalar and vector interactions leads to the mass-shell equation in the form of a linear relation between the canonical generators of the Lie algebra $so(2, 1)$. Demanding the preservation of this algebraic structure we have constructed a quantum mechanical description. In such a pure algebraic way we have obtained the mass spectrum which in particular cases agrees with the one-particle problem in an appropriate external field.

A more difficult problem of the quantization of the time-symmetric Wheeler–Feynman theory still remains unsolved. We only note that within the linear approximation in the coupling constant both theories coincide, and the mass spectrum (5.13) in the electromagnetic case ($\alpha_1 = 1, \alpha_0 = 0$) agrees perfectly with the results obtained within the framework of various more or less consistent methods from quantum electrodynamics in the several papers [49–51] (see also [39, 49] for the case of scalar interaction).

Particular attention has been paid to the question of the relation between the relativistic wave equation [1, 2] and our description. In the rest frame this equation also takes the form of a linear relation between generators of $so(2, 1)$ so that the vector of the algebra has the same Killing form as in our case. As a result, in the case of the vector interaction both treatments lead to the same mass spectrum. Both equations are related by the transformation from $SO(2, 1)$. All the coefficients in our mass-shell equation are immediately obtained from the Lagrangian. This made possible the generalization of Barut's wave equation for other interactions besides the electromagnetic one. The obtained values of coefficients A_1, A_2, A_3, B and D in equation (6.3) follow from the algebraic quantization of the Fokker-type action for the linear superposition of the scalar and vector time-asymmetric interactions. Moreover, all the equations of the type (6.3) which lead to the same Killing form (4.14) will give the same mass spectrum. We do not consider the possibility of further generalizations of equation (6.3) allowing arbitrary coefficients (cf [5]). Although infinite-component wave equations like (6.3) have their own physical problems (tachyonic states and so on, see, for example, [48]), we consider these equations as technical tools for constructing mass spectra and eigenstates of the given two-body problem.

The perfect agreement of our mass spectra with the results of other relativistic formalisms stimulated us to construct the classical relativistic Hamiltonian model which corresponds to the two-particle system in $\mathbb{M}_4$. In the four-dimensional spacetime the problem of the construction of a canonical realization of the Poincaré group supplemented by the spacetime interpretation in the front form of dynamics [26] is a much more complicated and ambiguous task than in the two-dimensional Minkowski space. Here we use the approach developed by Duviryak in [17], which allows one to construct the complete Hamiltonian description including the covariant particle worldlines within the Bakamjian–Thomas-like canonical realization of $\mathcal{P}(3, 1)$. In our classical relativistic two-particle Hamiltonian model the mass-shell equation has the same algebraic structure as the corresponding equation in the two-dimensional variant of the front form. Therefore, the quantization of the inner motion will give mass spectra (5.13) and (6.7) with a non-zero value of orbital quantum number.

The choice of the relation between the canonical variables determining the canonical realization of $\mathcal{P}(3, 1)$ and the covariant particle coordinates in the form (6.15) is caused by the connection of the mass-shell equation for the scalar–vector interaction ($\alpha_0^2 = \alpha_1^2$) with the time-asymmetric Fokker-type action integral in $\mathbb{M}_4$ [25]. The question of the correspondence between the constructed Hamiltonian model with arbitrary values of α_0, α_1 and the Fokker-type action in $\mathbb{M}_4$ still remains an open problem. However, one can show that in the second-order approximation in the coupling constant such a relation really does exist [52].

As we have seen the dynamical group $SO(2, 1)$ arises in the various treatments (non-relativistic case, infinity-component relativistic wave equation, Fokker-type action integrals) connected with the two-particle Coulomb problem. Therefore, it is very important to understand whether the appearance of this dynamical group is caused only by the particular formalism of relativistic mechanics (and therefore by a certain 'simplification' of the field theory) or whether it is typical of a two-particle system with particular types of field interactions. The existence of such an algebraic structure of the mass-shell equation plays a prominent role on the quantum level and enables us to obtain mass spectra without an explicit realization of operators in certain Hilbert space. To obtain a complete quantum-mechanical description it is necessary to construct a representation for which functions of the form (5.7) are orthonormal and span some Hilbert space of our problem. The role and importance of the dynamical group on the classical level are still not clearly understood. It poses the challenging problem of dynamical equivalence of the classical systems described by mass-shell equations which contain different elements of the Lie algebra $so(2, 1)$ but have the same Killing form.

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